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Research Note

Parametric Formulas for Geodesic Curves and Distances on a Slightly Oblate Earth

E.A. LEWIS

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Abstract

Approximate expressions for geodesic curves and the geodesic arc-lengths are obtained by straightforward methods which permit upper bounds of error to be established analytically. The errors are typically less than 1.4 parts per million, and even higher accuracy is possible with additional corrections.

Selected numerical examples are given, and calculated arc-lengths are compared with values obtained with Andoyer's approximate formula.

Contents

Parametric Formulas for Geodesic Curves and Distances on a Slightly Oblate Earth

1. INTRODUCTION.

Hyperbolic Direction Finders at Very Low Frequencies combine relatively high accuracy,[1] and operating ranges comparable to, and perhaps even larger than, the earth's radius. It is therefore important for computing lines of position to inquire what allowance should be made for the fact that the earth is more nearly an oblate spheroid than a true sphere. If it is assumed that the "first-to-arrive" components of an electro-magnetic pulse travel from the source to the receiver by the shortest possible surface route, and if it is further assumed that the velocity of propagation is constant along this path, then the problem of computing ray trajectories and travel-times is equivalent to mathematically calculating geodesics and geodesic arc-lengths.

As a result of the considerable attention which has been devoted to the geodesic problem, several rather elegant approximate solutions are already available.[2] In most cases, the accuracy of these solutions is high; but the limits of error are somewhat obscure.

The treatment in the following pages is straightforward to the point of being

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elementary; the condition for minimum path-length between two arbitrary points on the spheroid leads directly to a problem in the calculus of variations, which is then converted to a differential equation in spherical-polar coordinates θ and ϕ. This equation is easily solved to a high degree of precision in terms of an "adjusted" co-latitude angle ζ, thus giving the equation of the geodesics in parametric form. Geodesic arc-lengths are also obtained in parametric form by approximate integration along the geodesic curve. The analytic approximations used are shown to be better than 1.4 parts in a million, and are thus more than adequate for purposes of very long range radio location at the present state-of-the-art. Furthermore, the approximations are of such simple nature that if desired, an even higher accuracy can be obtained in numerical cases.

2. THE FIGURE OF THE EARTH, AND LATITUDE CONVERSION FORMULAS

Precision surveying and mapping techniques refer all latitudes and longitudes to a reference spheroid which has been chosen to approximate the figure of the earth, but whose placement and dimensions are to a certain extent arbitrary. Such a frame of reference constitutes a geodetic "datum". In the United States the North American Datum of 1927 is employed in modern work. In other countries other datums are used, but the problem of converting coordinates in one datum to those in another is beyond the scope of these considerations, which assume that the North American Datum is extendable over the whole earth, and represents the shape of the earth with sufficient accuracy.

In this datum, the reference geoid is the "Clarke Spheroid" of 1866 whose dimensions are listed in Table I,[3] along with certain derived constants used in the analysis to follow.

In rectangular coordinates x, y, z the equation of the spheroid is

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{1}$$

where the minor axis of the spheroid is taken to coincide with the OZ axis of coordinates. In the corresponding polar-spherical coordinates (R, θ, ϕ), the equation is:

$$\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{R^2} \tag{2}$$

Hence,

$$R = \frac{a}{\sqrt{1 + \delta \ \cos^2\theta}} \tag{3}$$

Table I. List of Constants

Semi Major Axis (a)	= 6378.2064 km
Semi Minor Axis (b)	= 6356.5838 km
Δ	$= a - b =$ 21.6226 km
$\frac{\Delta}{a}$	$= \frac{a - b}{a} =$ 0.00339008
$\frac{a + b}{2}$	= 6367.3951 km
$\frac{b}{a}$	= 0.99660992
$\left(\frac{b}{a}\right)^2$	= 0.9932 3134
$\left(\frac{b}{a}\right)^4$	= 0.98650849
$\frac{a}{b}$	= 1.0034016
$\left(\frac{a}{b}\right)^2$	= 1.0068148
δ	$\equiv \frac{a^2}{b^2} - 1 =$ 0.0068148
U	$\equiv \frac{\delta}{2} - \frac{3\delta^2}{8} =$ 0.0033900

Referring to Figure 1, the geodetic latitude of the point P is denoted φ . (This symbol is not to be confused with ϕ, the azimuthal angle in the polar co-ordinate system.) The Y O Z plane cuts the spheroid in the ellipse

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{4}$$

whence it follows that

$$\frac{dz}{dy} = -\frac{b^2}{a^2}\,\frac{y}{z} = -\frac{b^2}{a^2}\cot\varphi' \tag{5}$$

where φ' is the geocentric latitude. But,

$$\frac{dz}{dy} = \tan(\pi/2 + \varphi) = -\cot\varphi \tag{6}$$

and hence

$$\tan\varphi = \frac{a^2}{b^2}\tan\varphi' = \frac{a^2}{b^2}\cot\theta \tag{7}$$

Also,

$$\cos\theta = \frac{b^2}{\sqrt{a^4\cot^2\varphi + b^4}} \tag{8}$$

$$\sin\theta = \frac{a^2}{\sqrt{a^4 + b^4\tan^2\varphi}} \tag{9}$$

It follows from Eq. (3) that

$$R = a\sqrt{\frac{1 + \frac{b^4}{a^4}\tan^2\varphi}{1 + \frac{b^2}{a^2}\tan^2\varphi}} \tag{10}$$

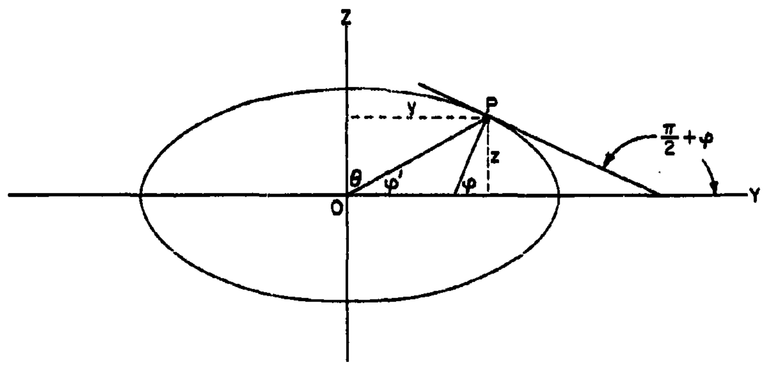

Figure 1. Section of Spheroid by YZ Plane

In the analysis to follow, much use is made of the parameter ζ defined by the relation:

$$\sin \zeta \equiv \frac{\sin \theta}{\sqrt{1 + \delta \cos^2 \theta}} \tag{11}$$

From this,

$$\cos \zeta = \sqrt{1 + \delta}\, \frac{\cos \theta}{\sqrt{1 + \delta \cos^2 \theta}} \tag{12}$$

and

$$\tan \zeta = \frac{\tan \theta}{\sqrt{1 + \delta}} = \frac{b}{a} \tan \theta \tag{13}$$

The inverse relations are:

$$\tan\theta = \sqrt{1+\delta}\ \tan\zeta = \frac{a}{b}\tan\zeta \tag{14}$$

$$\sin\theta = \frac{\sqrt{1+\delta}}{\sqrt{1+\delta\ \sin^2\zeta}}\ \sin\zeta \tag{15}$$

$$\cos\theta = \frac{\cos\zeta}{\sqrt{1+\delta\ \sin^2\zeta}} \tag{16}$$

and, using Eq. (3),

$$R = \frac{a}{\sqrt{1+\delta}}\ \sqrt{1+\delta\ \sin^2\zeta} = b\sqrt{1+\delta\ \sin^2\zeta} \tag{17}$$

Differentiating Eq. (14),

$$\frac{1}{\cos^2\theta}\ \frac{d\theta}{d\zeta} = \frac{\sqrt{1+\delta}}{\cos^2\zeta} \tag{18}$$

and by Eq. (16),

$$\frac{d\theta}{d\zeta} = \frac{\sqrt{1+\delta}}{1+\delta\ \sin^2\zeta} = \frac{a/b}{1+\delta\ \sin^2\zeta} \tag{19}$$

Finally it is often desirable to convert directly from ζ to φ, and vice versa, without using θ. In view of Eqs. (7) and (14), the needed relations are:

$$\tan\varphi = \frac{a}{b}\cot\zeta \tag{20}$$

$$\cos \varphi = \frac{1}{\sqrt{1 + \frac{a^2}{b^2} \cot^2 \zeta}} \tag{21}$$

$$\sin \varphi = \frac{1}{\sqrt{1 + \frac{b^2}{a^2} \tan^2 \zeta}} \tag{22}$$

The corresponding inverse relations are:

$$\tan \zeta = \frac{a}{b} \cot \varphi \tag{23}$$

$$\cos \zeta = \frac{1}{\sqrt{1 + \frac{a^2}{b^2} \cot^2 \varphi}} \tag{24}$$

$$\sin \zeta = \frac{1}{\sqrt{1 + \frac{b^2}{a^2} \tan^2 \varphi}} \tag{25}$$

The two sets of equations are symmetric.

The qualitative nature of the angles ζ and $\frac{\pi}{2} - \varphi$ is illustrated in exaggerated scale in Figure 2. It will be noted that either

$$\theta \geqslant \zeta \geqslant \frac{\pi}{2} - \varphi \qquad \text{or} \qquad \theta \leqslant \zeta \leqslant \frac{\pi}{2} - \varphi \tag{26}$$

3. THE DIFFERENTIAL EQUATION OF A GEODESIC, AND ITS SOLUTION

Consider a surface defined in spherical coordinates by the function $R = R(\theta)$. Then an arbitrary surface curve joining two points P_1 and P_2 on the surface can be defined by specifying ϕ as a function of θ. The curvilinear

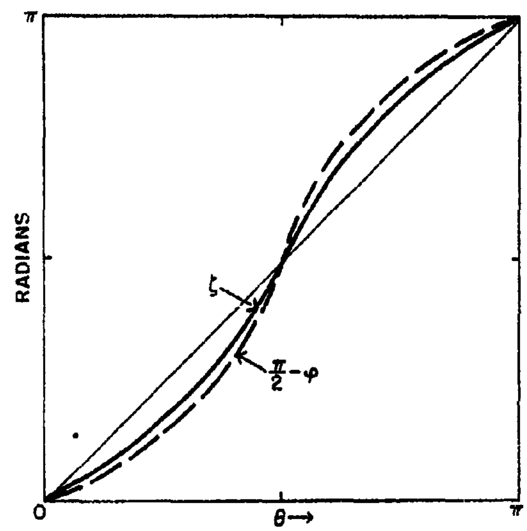

Figure 2. Angles ζ and $\pi/2 - \varphi$ as Functions of θ

distance S between P_1 and P_2 is then

$$S = \int_{P_1}^{P_2} I \, d\theta \tag{27}$$

where

$$I \equiv I\left(\theta, \frac{d\phi}{d\theta}\right) \equiv \sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2 + R^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2} \tag{28}$$

It is now a problem in the Calculus of Variations to chose $\phi(\theta)$ so that S will be a minimum. The desired function will be one of the extremals satisfying the Euler condition, which, since I does not contain ϕ explicitly is simply:

$$\frac{d}{d\theta} \frac{\partial I}{\partial\left(\frac{d\phi}{d\theta}\right)} = 0 \tag{29}$$

On integrating once,

$$\frac{\partial I}{\partial \left(\frac{d\phi}{d\theta}\right)} = \text{constant} \tag{30}$$

In view of Eq. (28), this becomes:

$$\frac{R^2 \sin^2 \theta \ \frac{d\phi}{d\theta}}{\sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2 + R^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} = \text{constant} \tag{31}$$

whence,

$$\frac{d\phi}{d\theta} = -\frac{C}{\sin\theta}\sqrt{\frac{1 + 1/R^2 \left(\frac{dR}{d\theta}\right)^2}{\left(\frac{R}{a}\right)^2 \sin^2\theta - C^2}} \tag{32}$$

where C is a new constant. In order to geometrically interpret this constant, a short digression is now appropriate.

In Figure 3 an elemental portion P_1G of a geodesic from point P_1 is shown. The corresponding elemental components along the meridian and along the minor circle are designated P_1V and P_1W respectively. The angle VP_1G is designated B, and is the bearing angle which the geodesic makes with respect to north. In the limit,

$$\tan B = \frac{\overline{P_1 W}}{\overline{P_1 V}} \tag{33}$$

Now,

$$\overline{P_1 W} = R \sin\theta \ d\phi$$

$$\overline{P_1 V} = \sqrt{R^2 \, d\theta^2 + d R^2}$$

$$= -R\sqrt{1 + \frac{1}{R^2}\left(\frac{dR}{d\theta}\right)^2} \quad d\theta$$

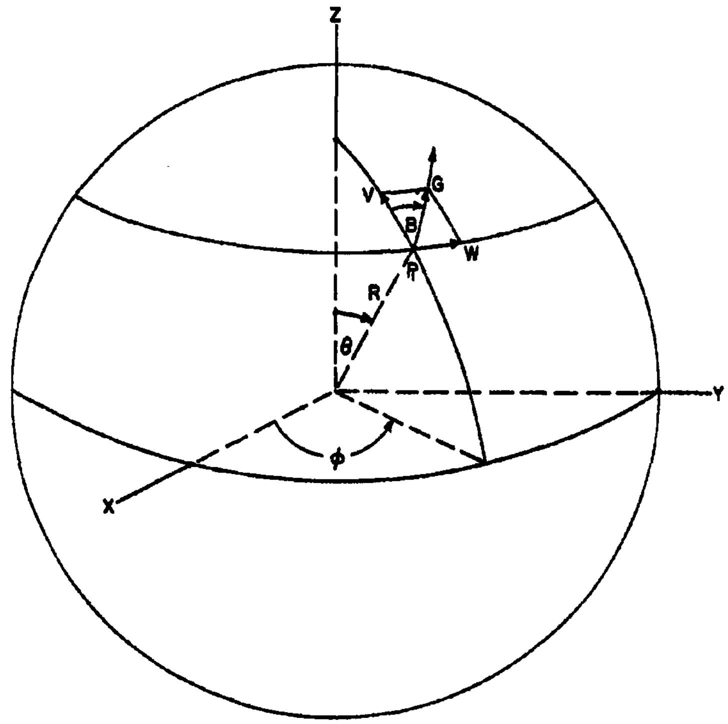

Figure 3. An Element of a Geodesic

In this, the negative root is chosen so that $-d\,\theta$, and hence $\overline{P_1 V}$, are positive. It follows that:

$$\tan B = -\frac{\sin\theta}{\sqrt{1+\frac{1}{R^2}\left(\frac{dR}{d\theta}\right)^2}}\;\frac{d\phi}{d\theta} \tag{34}$$

On making use of Eq. (32),

$$\tan B = \frac{C}{\sqrt{\left(\frac{R}{a}\right)^2 \sin^2\theta - C^2}} \tag{35}$$

$$= \frac{C}{\sqrt{\sin^2 \zeta - C^2}} \tag{36}$$

the second step being a consequence of Eqs. (3) and (11). On re-arranging Eq. (36), it will be found that

$$C^2 = \sin^2 \zeta \sin^2 B \tag{37}$$

In order to avoid ambiguities of sign, the convention will be adopted that, at the starting point P_1 of the geodesic, the positive roots of all the radicals are to be taken. It is then clear from Eq. (35) for example, that the sign of C is to be taken to be the same as the sign of tan B.

If B_{eq} is the bearing of the geodesic at $\zeta = 90°$ (that is, at the equator), it follows that

$$C^2 = \sin^2 B_{eq} \tag{38a}$$

At the "turn-around" point of the geodesic (in B = 0)

$$C^2 = \sin^2 \zeta_{N,S} \tag{38b}$$

where $\zeta_{N,S}$ represents the closest approach of the geodesic to the poles, and is two valued. (For $-\frac{\pi}{2} < B_{eq} < \frac{\pi}{2}$, the angles B_{eq} and ζ_N are equal.)

It is now seen that the constant C rather simply determines both the bearing of the geodesic at the equator, and the closest approach to the poles.

Returning now to the differential Eq. (32), the next step is to change from the variable θ to the variable ζ [see Eq. (11)]. Then

$$d\phi = - \frac{C\,A}{\sin \zeta \sqrt{\sin^2 \zeta - C^2}}\, d\zeta \tag{39}$$

where

$$A = \frac{\sqrt{1+\delta \sin^2 \zeta}}{\sqrt{1+\delta}} \sqrt{1+\left(\frac{1}{R}\ \frac{dR}{d\theta}\right)^2}\ \frac{d\theta}{d\zeta}$$

$$= \sqrt{\frac{1+\left(\frac{1}{R}\ \frac{dR}{d\theta}\right)^2}{1+\delta \sin^2 \zeta}} \tag{40}$$

thanks to Eq. (19). Next, on differentiating Eq. (3) with respect to θ,

$$\frac{1}{R}\ \frac{dR}{d\theta} = \frac{\delta \sin\theta\ \cos\theta}{1+\delta \cos^2\theta}$$

$$= \frac{\delta}{\sqrt{1+\delta}} \sin\zeta\ \cos\zeta \tag{41}$$

in view of Eqs. (15) and (16). Thus,

$$A = \sqrt{\frac{1+\frac{\delta^2}{1+\delta} \sin^2\zeta \cos^2\zeta}{1+\delta \sin^2\zeta}} \tag{42}$$

Consider now the new quantity

$$A' \equiv 1 - u \sin^2 \zeta \tag{43}$$

where

$$u \equiv \frac{\delta}{2} - \frac{3\delta^2}{8} \tag{44}$$

It may now be shown after some algebraic manipulation that

$$\frac{A'}{A} = \sqrt{1 - 2\digamma} \tag{45}$$

where

$$\digamma = \frac{\alpha}{16} \left[1 + \delta + \delta^2 \alpha (1 - \alpha) \right]^{-1} \left[2 \delta^2 (1 - \alpha) - \delta^3 (2 \alpha^2 - 3\alpha + 6) - \delta^4 \alpha \left(\frac{33}{8} - \alpha \right) - \frac{\delta^5}{8} \alpha (9 - 15\alpha) - \frac{9}{8} \delta^6 \alpha^2 \right] \tag{46}$$

in which for brevity,

$$\alpha \equiv \sin^2 \zeta \tag{47}$$

Now, by Table I,

$$\delta \approx 0.0068148$$

and hence:

$$\begin{aligned} \delta^2 &\approx 4.64415 \times 10^{-5} \\ \delta^3 &\approx 3.16490 \times 10^{-7} \\ \delta^4 &\approx 2.15681 \times 10^{-9} \\ \delta^5 &\approx 1.4698 \times 10^{-11} \\ \delta^6 &\approx 1.0016 \times 10^{-13} \end{aligned} \tag{48}$$

Hence it is clear that $\digamma$ is very small, and that the error in substituting A' for A will be approximately $\digamma$. It may be seen from Figure 4 that $\digamma$ is everywhere less than about 1.4×10^{-6}. Thus, to a very high degree of precision Eq. (39) can be written

$$d\phi \approx - C \frac{1 - u \sin^2 \zeta}{\sin \zeta \sqrt{\sin^2 \zeta - C^2}} \, d\zeta \tag{49}$$

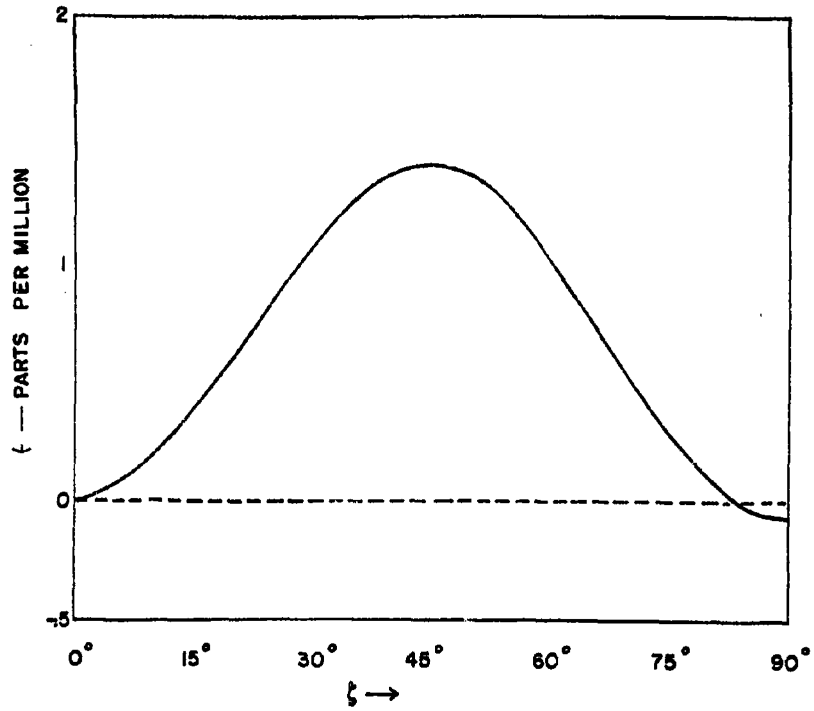

Figure 4. ƒ as a Function of ξ

It will be noted from Eq. (39) that $C^2 \leqslant 1$. Inspection of the denominator of Eq. (49) shows that at least for $C \neq 0$, the geodesic cannot reach a latitude higher than that corresponding to $\sin^2 \zeta = C^2$ since beyond this point the coefficient of $d\,\zeta$ becomes imaginary. (See also Eq. (38b).) If ϕ is to continue to increase "beyond" the "turn-around" point, the radical in Eq. (49) must change sign at that point. If $d\,\phi\,/\,d\,\zeta$ and ζ were plotted for a procession of points following along a geodesic which starts at the point P_1 in the Northern Hemisphere, and runs in a more or less North-Easterly direction, the result would be similar to that depicted in Figure 5. There, starting at P_1, ζ is decreasing and $d\,\phi\,/\,d\,\zeta$ is negative and decreasing, going to "minus infinity" at the turn-around point T_1. Thereafter ζ increases, while $d\,\phi\,/\,d\,\zeta$ is positive and decreasing (upper branch in Figure 5). A minimum is reached at $\zeta = \frac{\pi}{2}$, and then $d\,\phi\,/\,d\,\zeta$ goes to "plus infinity" at the second turning point, T_2, and so on.

If ϕ_1 and ζ_1 are the coordinates of P_1, and ϕ, ζ are the coordinates of a running point P on the geodesic, lying between P_1 and T_1, it follows from

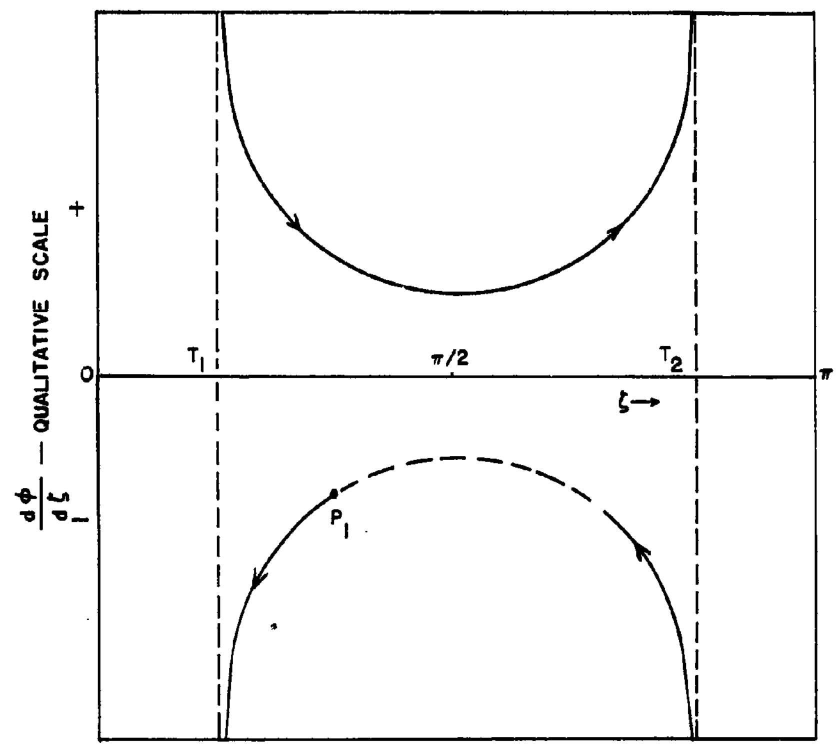

Figure 5. Derivative $\frac{d\theta}{d\zeta}$ as a Function of ζ

Eq. (49) that

$$\phi - \phi_1 \approx -C \int_{\zeta_1}^{\zeta} \frac{1 - u \sin^2 \zeta}{\sin \zeta \sqrt{\sin^2 \zeta - C^2}} \, d\zeta \qquad (50)$$

and this will be a positive quantity. If P lies on the far side of the turning point T_1, the corresponding equation is,

$$\phi - \phi_1 \approx -C \int_{\zeta_1}^{T_1} \frac{1 - u \sin^2 \zeta}{\sin \zeta \sqrt{\sin^2 \zeta - C^2}} \, d\zeta + C \int_{T_1}^{\zeta} \frac{1 - u \sin^2 \zeta}{\sin \zeta \sqrt{\sin^2 \zeta - C^2}} \, d\zeta \qquad (51)$$

where allowance was made for the change in sign of the radical. As might be expected it turns out that the result of integrating Eq. (50), namely

$$\phi - \phi_1 \approx \left[\overset{*}{\sin}^{-1}\left(\frac{C}{\sqrt{1 - C^2}} \cot \zeta\right) - C\,u\ \overset{*}{\sin}^{-1}\left(\frac{\cos \zeta}{\sqrt{1 - C^2}}\right)\right]_{\zeta_1}^{\zeta} , \qquad (52)$$

includes the second case as well, provided the functions $\overset{*}{\sin}^{-1}$ are taken to have their "principal values" in the first case (that is, $\zeta_1 < \zeta < T_1$), and the next larger values in the region $T_1 < \zeta < T_2$, and so on. The asterisks are used as a reminder to choose the correct range of the quantity so marked. To avoid any possibility of confusion, reference may be made to Table II.

Table II. Ranges of $\overset{*}{\sin}^{-1}$

Range of ζ	Range of $\overset{*}{\sin}^{-1}$
P_1 to T_1	0 to $\pi/2$
T_1 to T_2	$\pi/2$ to $3\pi/2$
T_2 to T_3	$3\pi/2$ to $5\pi/2$
T_3 to T_4	$5\pi/2$ to $7\pi/2$
etc.	etc.

Note: T_1, T_2, T_3, etc. are the successive "turn-around" points of the geodesic.

Except in the cases $C = 0$, the longitude difference $\phi - \phi_1$ continually increases as the running point P moves forward along the geodesic, the contributions of all parts of the integral $\int_{\zeta_1}^{\zeta}$ are of one sign, there being no possibility of one part cancelling, or tending to cancel, another. Under this condition it is justifiable to conclude that the maximum error in $\phi - \phi_1$, is less than 1.4 parts per million. Thus, even if $\phi - \phi_1$ is as large as 180°, the error in $\phi - \phi_1$ is less than 0.000252°, which at most would correspond to a positional error along a parallel of 34 meters. Of course, if necessary, the integral in Eq. (50), for example, could be broken up into sub-integrals and each corrected for the slight

difference between A and A^1, thus reducing the error to any desired limit.

Eq. (52) has an important simplification when the starting point P_1 of the geodesic is on the equator, for then the value of the indefinite integral at $\zeta = \zeta_1$ vanishes, and

$$\phi - \phi_1 \approx \overset{*}{\sin}^{-1}\left(\frac{C}{\sqrt{1 - C^2}} \cot\zeta\right) - C\,u\, \overset{*}{\sin}^{-1}\left(\frac{\cos\zeta}{\sqrt{1 - C^2}}\right) \tag{53}$$

In terms of the bearing angle B_{eq} at P_1 (See Eq. (36)):

$$\phi - \phi_1 \approx \overset{*}{\sin}^{-1}(\tan B_{eq} \cot\zeta) - u \sin B_{eq}\, \overset{*}{\sin}^{-1}\left(\frac{\cos\zeta}{\cos B_{eq}}\right) \tag{54*}$$

Eqs. (52), (53), and (54) for the geodesic may now be interpreted geometrically in terms of a reference sphere (Figure 6) on which the polar and azimuthal coordinates are ζ and ϕ respectively. In Figure 6, the point A is antipodal to P_1, and P_1P^1A is a great circle such that the angle Z P_1P^1 at P_1 is B_{eq}. Let $\Delta\phi^1$ be the azimuth of P^1 and ζ the polar angle.

In the spherical triangle Z P_1P^1, the "sine law" gives

$$\sin\zeta = \frac{\sin B_{eq} \sin\gamma}{\sin\Delta\phi^1} \tag{55}$$

γ being the angle P_1 0 P^1 . The "cosine law" applied to the same triangle gives

$$\sin\gamma = \frac{\cos\zeta}{\cos B_{eq}} \tag{56}$$

and,

$$\gamma = \overset{*}{\sin}^{-1}\left(\frac{\cos\zeta}{\cos B_{eq}}\right) \tag{57}$$

Eliminating $\sin\gamma$ from Eqs. (55) and (56), gives ,

*Equations (54) and (72) have been programmed for an IBM 1620 computer by the AFCRL Technical Services Division under contract AF 19(628)-411.

$$\sin \Delta\phi^1 = \tan B_{eq} \cot \zeta \qquad (58)$$

and,

$$\Delta \phi^1 = \overset{*}{\sin}{}^{-1} (\tan B_{eq} \cot \zeta) \qquad (59)$$

The geodesic, Eq. (54), can now be written

$$\phi - \phi_1 \approx \Delta \phi^1 - u \gamma \sin B_{eq} \qquad (60)$$

The "map" of the geodesic on the reference sphere may therefore be constructed by drawing a great circle, and then reducing the azimuthal angle of each point on it by $u\gamma \sin B_{eq}$, where γ is the angular measure of distance along the great circle. In Figure 6 the broken line P_1PP_2 represents the geodesic derived from the great circle.

Any geodesic from P_1 (on the equator) with $0 < B_{eq} < \frac{\pi}{2}$ will intersect the equator again at P_2 which falls West of the antipodal point A by the small arc-distance

$$\overline{P_2 A} \approx \pi u \sin B_{eq} \qquad (61)$$

(See Eq. (60).) The same geodesic, if continued through the southern hemisphere, will by symmetry intersect the equator again a distance $2\pi u \sin B_{eq}$, west of P_1, and so on. Unless the exact value of $\frac{\overline{P_2A}}{\pi}$ is a rational number, the geodesic will never close on itself but will continue to "creep" round and round the sphere. Figure 7 is intended to illustrate qualitatively the course of geodesic from the equatorial point P_1 as viewed from above the north pole of an earth with an exaggerated degree of flattening. The dotted portions of the curves represent parts of the geodesic in the southern hemisphere. The geodesic may be visualized as the curve generated by tightly winding a thread over the surface of the (frictionless) spheroid, starting at P_1 and passing through P_2.

In the special case $C = 0$ the geodesic starts out due North, and it is intuitive that it will follow a meridian. For this case, Eq. (53) correctly gives $\phi - \phi_1 = 0$.

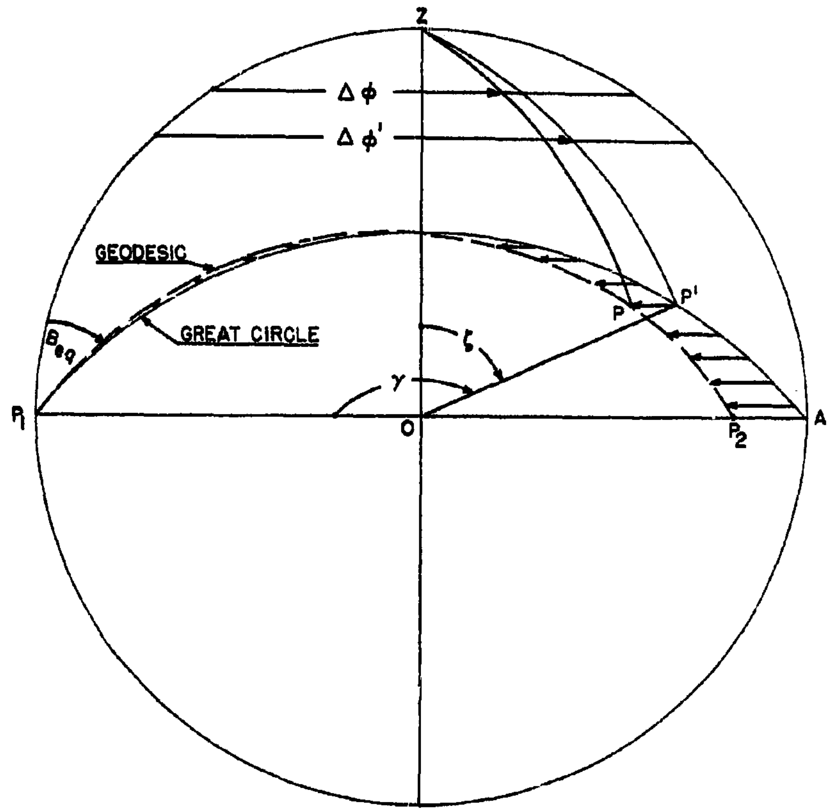

Figure 6. The Reference Sphere

Another special case of interest is when P_1 is on the equator, and $C \longrightarrow 1$, that is, the initial bearing approaches 90° which would take the curve on an eastward, equatorial course. In this case Eq. (53) is properly indeterminate, since on the equator ϕ may have any value.

Finally, there is an interesting characteristic of the geodesics in the vicinity of the point A antipodal to the equatorial starting point P_1. By Eq. (61) it will be seen that regardless of the equatorial bearing angle B_{eq}, the small equatorial arc P_2A (See Figure 7) can never exceed a value of approximately π u. This defines a sort of limiting point L_W such that the angular distance

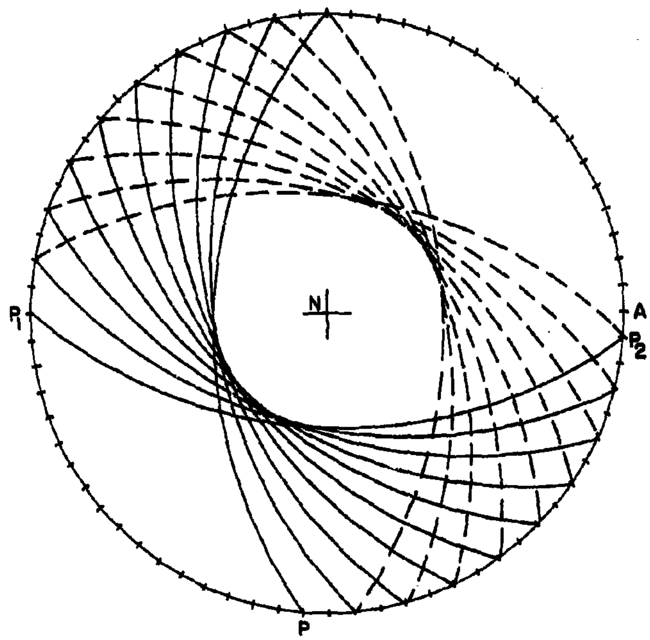

Figure 7. Multi-Turn Portion of a Geodesic

$$\overline{L_W A} \approx \pi u \tag{62}$$

and there is a similar point L_E on the east of the antipodal point. If P_2 is precisely at the antipodal point, the shortest route from P_1 to P_2 is precisely over a pole. If P_2 lies between L_W and A, or A and L_E, the shortest route is neither polar nor equatorial, but is an arc lying in either the north or southern hemispheres. However, if P_2 lies anywhere on the equator outside the small arc L_W A L_E , the shortest route will be precisely along the equator itself.

4. ARC LENGTH ON A GEODESIC

The arc length along a geodesic between the starting point P_1 and an arbitrary running point P will be found by performing the integration indicated in

Eq. (27), namely

$$S = \int_{P_1}^{P} ds = \int_{P_1}^{P} I \; d\theta \tag{63}$$

where in view of Eq. (32), Eq. (28) becomes

$$I = \sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2 + C^2 \; \frac{\left(\frac{dR}{d\theta}\right)^2 + R^2}{\frac{R^2}{a^2} \sin^2 \theta - C^2}} \tag{64}$$

$$= \frac{\frac{R^2}{a} \sin\theta \sqrt{1 + \left(\frac{1}{R} \; \frac{dR}{d\theta}\right)^2}}{\sqrt{\frac{R^2}{a^2} \sin^2 \theta - C^2}} \tag{65}$$

Using Eq. (3)

$$\frac{R^2}{a^2} = \frac{1}{1 + \delta \cos^2 \theta}$$

and hence by Eq. (11)

$$\frac{R^2}{a^2} \sin^2 \theta = \sin^2 \zeta$$

On squaring Eq. (17) and multiplying by Eq. (15),

$$R^2 \sin\theta = \frac{a^2}{\sqrt{1 + \delta}} \sqrt{1 + \delta \sin^2 \zeta}$$

On substituting, Eq. (65) becomes

$$I = a\sqrt{\frac{1 + \delta \sin^2 \zeta}{1 + \delta}} \sqrt{\frac{1 + \left(\frac{1}{R} \frac{dR}{d\theta}\right)^2}{\sin^2 \zeta - C^2}} \sin \zeta \tag{66}$$

Putting this in Eq. (63) and changing variable from θ to ζ

$$S = -a \int_{\zeta_1}^{\zeta} \frac{A \sin \zeta}{\sqrt{\sin^2 \zeta - C^2}} d\zeta \tag{67}$$

where

$$A = \sqrt{\frac{1 + \delta \sin^2 \zeta}{1 + \delta}} \sqrt{1 + \left(\frac{1}{R} \frac{dR}{d\theta}\right)^2} \frac{d\theta}{d\zeta}$$

is precisely the same quantity already encountered in the previous section, (see Eq. (40)) where it was shown that it could be replaced by the quantity A' of Eq. (43) with an error less than 1.4 parts per million.* (In Eq. (67) the minus sign was chosen to give a positive value of S when $\sqrt{\sin^2 \zeta - C^2}$ is regarded as starting out positive at $\zeta = \zeta_1$ as in Section 3, and again changing sign at the "turn-around" point.) Thus

$$S \approx -a \int_{\zeta_1}^{\zeta} \frac{(1 - u \sin^2 \zeta) \sin \zeta}{\sqrt{\sin^2 \zeta - C^2}} d\zeta \tag{68}$$

On making use of the identity:

*As mentioned previously in connection with the integral of Eq. (50), the range of integration can be broken into segments for each of which a correction can be applied for the slight difference between A and A^1, thus obtaining an even more precise approximation.

$$1 - u \sin^2 \zeta \;\equiv\; \left[1 - \frac{u}{2}\,(1 + C^2)\right] + \frac{u}{2}\left[1 + C^2 - 2\sin^2\zeta\right] \quad , \tag{69}$$

Eq. (68) becomes:

$$\begin{aligned} S \approx a \left[1 - \frac{u}{2}\,(1 + C^2)\right] \int_{\zeta_1}^{\zeta} \frac{-\sin\zeta \; d\zeta}{\sqrt{\sin^2\zeta - C^2}} \\ - \frac{a\,u}{2} \int_{\zeta_1}^{\zeta} \frac{1 + C^2 - 2\sin^2\zeta}{\sqrt{\sin^2\zeta - C^2}} \sin\zeta \; d\zeta \end{aligned} \tag{70}$$

$$= a \left[\left\{1 - \frac{u}{2}\,(1 + C^2)\right\} \overset{*}{\sin}{}^{-1}\left(\frac{\cos\zeta}{\sqrt{1 - C^2}}\right) - \frac{u}{2}\cos\zeta \; \overset{*}{\sqrt{\sin^2\zeta - C^2}}\right]_{\zeta_1}^{\zeta} \tag{71}$$

If the starting point of the geodesic is on the equator, $\zeta_1 = \frac{\pi}{2}$, and the indefinite integral vanishes at the lower limit, giving simply

$$S \approx a \left[\left\{1 - \frac{u}{2}\,(1 + C^2)\right\} \overset{*}{\sin}{}^{-1} \frac{\cos\zeta}{\sqrt{1 - C^2}} - \frac{u}{2}\cos\zeta \; \overset{*}{\sqrt{\sin^2\zeta - C^2}}\right] \tag{72}$$

In the last two equations, the asterisks (as before) are reminders to change from "principal values" to the appropriate branch. In this connection it is noted that in the first integral of Eq. (70), the integrand is essentially positive since at the start, $d\zeta$ is negative, and later when the radical changes sign, so also does $d\zeta$. In the second integral, the integrand

$$Q \;\equiv\; \frac{1 + C^2 - 2\sin^2\zeta}{\sqrt{\sin^2\zeta - C^2}} \sin\zeta \tag{73a}$$

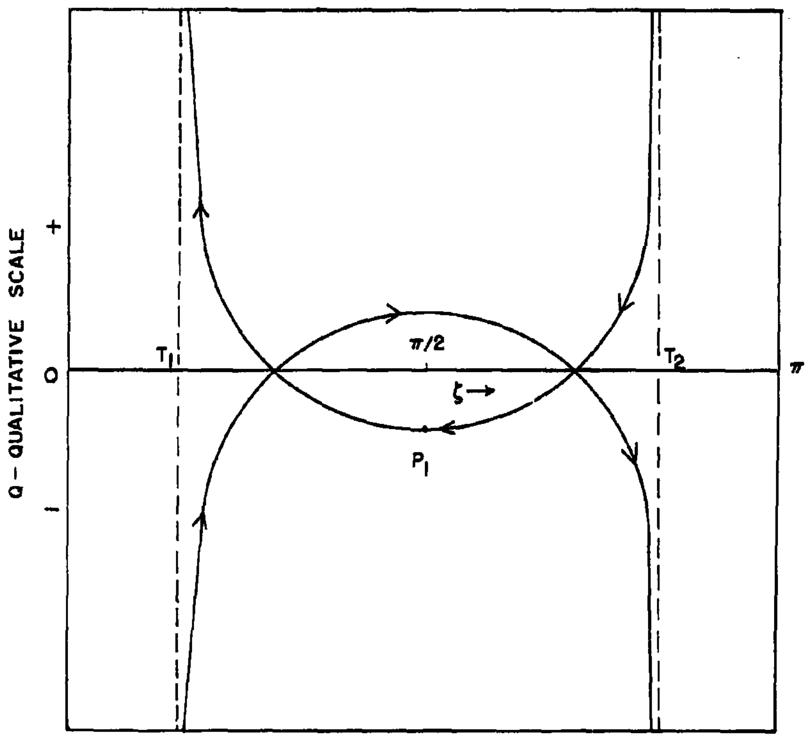

Figure 8. Qualitative Behavior of Integrand "Q"

has a behavior of the type illustrated qualitatively in Figure 8. Starting at P_1 on the equator, $Q = -\sqrt{1 - C^2}$. Then following along the geodesic Northerly and Easterly, Q increases to zero when $\sin \zeta = \frac{1 + C^2}{2}$, and then approaches $+\infty$ as the turning point of the geodesic is reached. Here the radical changes sign, and Q increases from $-\infty$, and so on.

It follows from these considerations that $\int_{\zeta_1}^{\zeta} Q \, d\zeta$ must have a behavior somewhat as shown in Figure 9. Since when starting out along the geodesic from P_1, $d\zeta$ is negative, the integral at first increases, reaching a maximum when $\sin \zeta = \frac{1 + C^2}{2}$, and then decreases to zero at the first turning

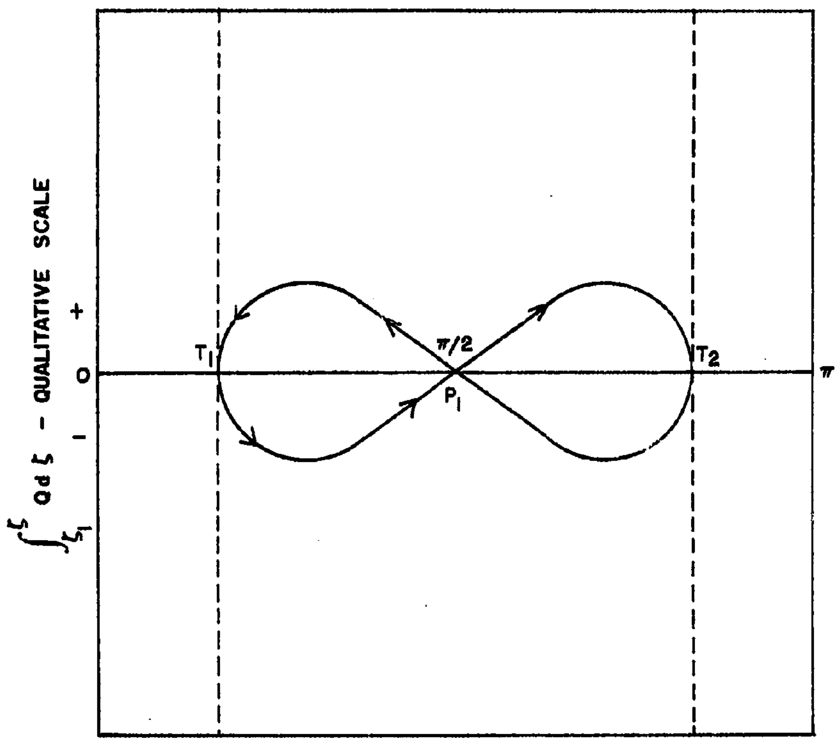

Figure 9. Qualitative Behavior of $\int_{\zeta_1}^{\zeta} Q\ d\ \zeta$

point T_1. Proceeding further along the geodesic, Q is negative and $d\zeta$ is positive, so that the integral goes negative, reaching a minimum when $\sin\zeta = \frac{1 + C^2}{2}$. It then increases through zero (when the geodesic crosses the equator) and so on. Evidently then, the sign of the quantity $*\sqrt{\sin^2\zeta - C^2}$ is to be changed at each reflection point. In Eq. (72) the factor $\cos\zeta$ automatically takes care of the sign change at the equator.

In Eq. (72), as already known from Section 3, the quantity $\sin^{-1}\left(\frac{\cos\zeta}{\sqrt{1 - C^2}}\right)$ is simply the arc $\overline{P_1P^1}$ (in radian measure), on the reference sphere, Figure 6. The factors containing u are the "corrections" applied to this great circle distance to obtain S.

The difference in the two routes ,

$$S_{eq} - S \approx \frac{\pi}{2} a\,u\ (1 - \sin B_{eq})^2 , \ (0 \leqslant B_{eq} \leqslant \pi/2) \tag{78}$$

can be as large as $\frac{\pi}{2} a\,u$ or about 33.96 km.

<u>Shortest Distance Between Ordinary Points on the Equator</u>

For all pairs of points on the equator excluding those considered above, it is clear intuitively that

$$S = a\ (\Delta\ \phi\ ) \tag{79}$$

where $(\Delta \phi)$ is the difference in longitude in radians. On setting $C = 1$ in Eq. (72) an indeterminancy arises, but this can be avoided by regarding the equatorial arc P_1P_2 (See Figure 10)as the limiting case of the geodesic arc $P_1P_2^{\ 1}$, when the point $P_2^{\ 1}$ moves approximately along the meridian NP_2 to approach P_2 . Thus for the point $P_2^{\ 1}$, the ζ - value approaches $\pi/2$, and so does B_{eq} , so that in the limit, Eq. (54) for the geodesic curve becomes:

$$\Delta\ \phi \longrightarrow \overset{*}{\sin}{}^{-1} \left(\frac{\cos \zeta}{\cos B_{eq}}\right) - u\ \overset{*}{\sin}{}^{-1} \left(\frac{\cos \zeta}{\cos B_{eq}}\right)$$

$$= (1 - u)\ \overset{*}{\sin}{}^{-1} \left(\frac{\cos \zeta}{\cos B_{eq}}\right) \tag{80}$$

Thus if ζ and B_{eq} are varied so that

$$k \equiv \frac{\cos \zeta}{\cos B_{eq}} \tag{81}$$

is a constant, the point $P_2^{\ 1}$ will approach P_2. Then since $C^2 = \sin^2 B_{eq}$ by

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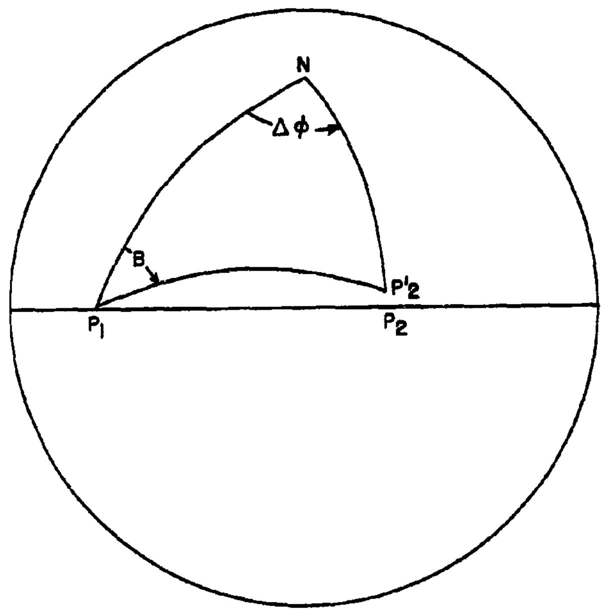

Figure 10. Limiting Process for Two Points on the Equator

Eq. (38a), it follows from Eqs. (72) and (80) that

$$S \rightarrow a\,(1 - u)\,\overset{*}{\sin}^{-1}\;k = a\,(\Delta\,\phi) \tag{82}$$

and this is the same as Eq. (79), as expected.

5. NUMERICAL EXAMPLES

5.1. Meridian Quadrant

As a first example, the length of the meridian quadrant will be calculated with the formula of Section 4, using the Bessel Spheroid,[4] for which the meridian quadrant is 10,000.8557658 km. The semi-major and semi-minor axes are

$$a = 6377.39715500 \text{ km}$$

$$b = 6356.07896325 \text{ km}$$

whence it follows that

$$\delta = 0.006719218$$

and

$$u = 0.0033426786.$$

By Eq. (74)

$$S_{mq} \approx \frac{\pi a}{2}\left(1 - \frac{u}{2}\right)$$

$$= 10{,}000.84923 \qquad (83)$$

This differs from the correct value by only about 6.5 meters or about 6.5 parts in ten-million.

5.2. Geodesic at B = 45° from Point on the Equator of Clarke Spheroid

For this

$$C = \sin B_{eq} = \frac{1}{\sqrt{2}}$$

and Eq. (51) for the geodesic becomes:

$$\phi - \phi_1 \approx \overset{*}{\sin}{}^{-1} (\cot \zeta) - \frac{u}{\sqrt{2}} \overset{*}{\sin}{}^{-1} (\sqrt{2} \cos \zeta) \tag{84}$$

5.2.1. At the point where the geodesic reaches latitude $\zeta = 60°$ beyond the first turning point, this equation gives

$$\phi - \phi_1 \approx 144.7356° - \frac{u}{\sqrt{2}} 135° \tag{85}$$

$$\approx 144.4120°$$

Latitude $\zeta = 60°$ corresponds to geodetic latitude $\varphi = 30.0843°$. By Eq. (72)

$$S \approx a \left[(1 - .75\, u)\, \overset{*}{\sin}{}^{-1} \left(\frac{1}{\sqrt{2}} \right) + \frac{u}{4} \sqrt{\frac{3}{4} - \frac{1}{2}} \right]$$

$$\approx a \left[\frac{0.9974575 \times 135°}{57.2957795} + 0.00042375 \right]$$

$$\approx 2.3506276\ a$$

$$\approx 14{,}992 \cdot 788 \text{ km}$$

5.2.2. At the point where the geodesic reaches latitude $\zeta = 60°$ <u>before</u> the first turning point, Eq. (84) gives

$$\phi - \phi_1 \approx 35.2644° - \frac{u}{\sqrt{2}} 45° \tag{86}$$

$$\approx 35.1565°$$

By Eq. (72)

$$S \approx a\left[(1-.75\,u)\ \overset{*}{\sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{u}{4}\ \sqrt{\frac{3}{4}-\frac{1}{2}}\right]$$

$$\approx a\left[\frac{0.9974575\times 45^\circ}{57.2957795} - 0.00042375\right]$$

$$\approx a\ [0.78340129 - 0.00042375]$$

$$\approx 0.7829775\ a$$

$$\approx 4993.992,\ \mathrm{km}$$

5.2.3. Length of geodesic arc on the previous curve, between points where $\zeta = 60^\circ$ ($\varphi = 30.0834^\circ$) is simply

$$14{,}992.788 - 4{,}993.992 = 9{,}998.796\ \mathrm{km}.$$

6. NOTE ON ANDOYER'S FORMULA

For calculating geodesic arc-lengths, use is often made of a formula due to Andoyer,[5] which, in the present notation, is

$$S \approx a\sigma + \delta S \tag{87}$$

where σ is the great-circle distance between two points computed as if the earth were a sphere; that is,

$$\cos\sigma \equiv \sin\varphi_1\ \sin\varphi_2 + \cos\varphi_1\ \cos\varphi_2\ \cos(\phi_1 - \phi_2) \tag{88}$$

and

$$\delta S \equiv p\ (\sin\varphi_1 + \sin\varphi_2)^2 - q\ (\sin\varphi_1 - \sin\varphi_2)^2 \tag{89}$$

$$p \equiv \frac{a-b}{8} \quad \frac{3\sin\sigma - \sigma}{\cos^2\frac{\sigma}{2}} \tag{90}$$

$$q \cong \frac{a-b}{8} \quad \frac{3\sin\sigma + \sigma}{\sin^2\frac{\sigma}{2}} \tag{91}$$

For two points on the equator, $\varphi_1 = \varphi_2 = 0$, and $\delta S = 0$. (The case when $\sigma = \pi$, for which p is infinite will be excluded from the present consideration.) Then by Eq. (87),

$$S \approx a\sigma \tag{92}$$

This is evidently the correct answer as far as the equatorial route between P_1 and P_2 is concerned. However, as discussed in Section 4, if the longitude ϕ_2 of P_2 is such that

$$\left[\phi_1 + \pi\right] > \phi_2 > \left[\phi_1 + \pi\ (1-u)\right] \tag{93}$$

the sub-polar route is shorter than the equatorial route by the amount [see Eq. (78)] ,

$$S_{eq} - S \approx \frac{\pi a u}{2}\ (1 - \overset{*}{\sin} B)^2 \tag{94}$$

which can be almost as much as 33.96 km. Thus in such cases, the Andoyer Formula, while still giving a "geodesic distance", gives the longer rather than the shorter extremal. In view of these particular arcs of nearly 180° for which Andoyer's formula fails, it is of interest to compare the results of Andoyer's formula with those obtained by the use of Eqs. (71) or (72) in Examples 5.2, 5.2.1, and 5.2.2. It will be seen that the agreement is excellent in the following examples.

6.1. **Distance between** $(\phi_1 = 0_1\ \varphi_1 = 0)$ **and** $(\phi_2 = 144.4120°, \varphi_2 = 30.0843)$.

This corresponds to Example 5.2, where it was found that S ≈ 14,922.789 km. Now,

$$\cos\sigma = \cos 30.0843° \ \cos 144.4120° = -0.70367249$$

$$\sigma = 134.7224° = 2.351349 \text{ radians}$$

$$3\sin\sigma = 3 \times 0.7105245 = 2.131573_5$$

$$\sin^2\frac{\sigma}{2} = \frac{1}{2}(1-\cos\sigma) = 0.85183625$$

$$\cos^2\frac{\sigma}{2} = \frac{1}{2}(1+\cos\sigma) = 0.14816376$$

$$p = \frac{a-b}{8}\ \frac{2.131573 - 2.351349}{0.14816376} = -\frac{a-b}{8}\ 1.48333169$$

$$q = \frac{a-b}{8}\ \frac{2.131573 + 2.351349}{0.85183625} = \frac{a-b}{8}\ 5.26265700$$

$$\delta S = (p-q)\sin^2\varphi = -\frac{a-b}{8}\sin^2 30.0843° \ (1.48333169 + 5.26265700)$$

$$= \frac{-21.6226}{8}\ (0.5012736)^2 \ 6.74598869$$

$$= -2.702\ 825 \times 0.2512752 \times 6.74598869$$

$$= -4.5815577$$

and

$$S \approx a\sigma + \delta S = 14{,}997.39 - 4.58 = 14{,}992.81 \text{ km.}$$

This is about 0.02 km longer than the value of Example 5.2.

6.2 Distance between $(\phi_1 = 0, \varphi_1 = 0)$ and $(\phi_2 = 35.1565°, \varphi_2 = 30.0843°)$

This corresponds to Example 5.2.1.

Now,

$$\cos\sigma = \cos 30.0843° \cos 35.1565° = 0.8652888 \times 0.8175822 = 0.70744472$$

$$\sigma = 44.9724° = 0.784916 \text{ radians}$$

$$3\sin\sigma = 3 \times 0.7067660 = 2.120298$$

$$\sin^2\frac{\sigma}{2} = \frac{1}{2}(1 - \cos\sigma) = 0.1462776$$

$$\cos^2\frac{\sigma}{2} = \frac{1}{2}(1 + \cos\sigma) = 0.8537224$$

$$p = \frac{a-b}{8} \quad \frac{2.120298 - 0.784916}{0.8537224} = \frac{a-b}{8} \quad 1.564188$$

$$q = \frac{a-b}{8} \quad \frac{2.120298 + 0.784916}{0.1462776} = \frac{a-b}{8} \quad 19.86096$$

$$\delta S = \frac{a-b}{8} \quad \sin^2 30.0843° \; (1.564188 - 19.86096)$$

$$= -\frac{21.6226}{8} \quad (0.5012736)^2 \; 18.29677$$

$$= -2.702825 \times 0.2512752 \times 18.29677$$

$$= -12.42630 \text{ km}$$

and

$$S \approx a\sigma + \delta S = 5006.356 - 12.426 = 4993.930 \text{ km}$$

This answer is slightly less (0.06 km) than that obtained in Example 5.2.1.

6.3 **Distance between** $(\phi_1 = 35.1565°, \varphi_1 = 30.0843°)$ **and** $(\phi_2 = 144.4120°, \varphi_2 = 30.0843°)$.

This corresponds to Example 5.2.2.
Now,

$$\cos\sigma = \sin^2 30.0843° + \cos^2 30.0843° \cos(144.4120° - 35.1565°)$$

$$= 0.2512752 + 0.7487247 \cos(109.2555°)$$

$$= 0.2512752 - 0.2469153$$

$$= 0.0043599$$

$$\sigma = 89.7502° = 1.566436 \text{ radians}$$

$$3\sin\sigma = 3(0.9999905) = 2.999971_5$$

$$\sin^2\frac{\sigma}{2} = \frac{1}{2}(1 - \cos\sigma) = 0.4978200_5$$

$$\cos^2\frac{\sigma}{2} = \frac{1}{2}(1 + \cos\sigma) = 0.5021799_5$$

$$p = \frac{a-b}{8} \quad \frac{2.999971_5 - 1.566436}{0.5021799_5} = 2.854625$$

$$q = \frac{a-b}{8} \; \frac{2.999971_5 + 1.568436}{0.4978200_5} = 9.172807_5$$

$$\delta S = 4\,p\,\sin^2 30.0843^\circ = \frac{21.6226}{2} \times 2.854625 \times 0.2512752$$

$$= 7.754907 \text{ km}$$

$$S \approx a\,\sigma + \delta S = 9991.052 + 7.755 = 9998.807 \text{ km}$$

This value is only 0.011 km greater than the value found in Example 5.2.2.

Acknowledgments

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Selected numerical examples are given, and calculated arc-lengths are compared with values obtained with Andoyer's approximate formula.

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